International Journal of Theoretical Physics, Vol. 28, No. 1, 1989

Sets of States and Extreme Points

Karsten Keller[1]

Received April 18, 1988

The natural embedding of orthoposets and quantum logics, equipped with certain sets of states, into their corresponding order-unit normed vector space is investigated. Necessary (resp. sufficient) conditions are stated for the case that the image of the embedding and the extreme points of the order interval, bounded by 0 and the order unit, coincide. Modifications of the state space are discussed from this point of view and the special case of a Boolean algebra is characterized.

1. INTRODUCTION

Let $(P, \leq, 0, 1)$ be a partially ordered set with a greatest element 1 and a least element 0. A unary operation $^\perp$: $P \to P$ is said to be an orthocomplementation if for all $p, q \in P$, (01) $p^{\perp\perp} = p$, (02) $\sup\{p, p^\perp\} = 1$, and (03) $p \leq q$ implies $q^\perp \leq p^\perp$. If $^\perp$ is an orthocomplementation, then $(P, \leq, ^\perp, 0, 1)$ is called an orthoposet. One says that $p, q \in P$ are orthogonal ($p \perp q$) if $p \leq q^\perp$.

$(P, \leq, ^\perp, 0, 1)$ is said to be an orthomodular poset or a quantum logic if for all $p, q \in$ P, (04) $p \leq q$ implies $\sup\{p, \inf\{p^\perp, q\}\}$. In this case one calls P $\perp$-complete (orthogonally complete) if each set of mutually orthogonal elements of P admits a supremum.

A state on P is defined to be a monotonically increasing positive real function μ on P with (S1) $\mu(1) = 1$, (S2) $\mu(\sup_{i=1}^n p_i) = \sum_{i=1}^n \mu(p_i)$ if $\{p_i\}_{i=1}^n$ is a family of mutually orthogonal elements of P, for which $\sup_{i=1}^n p_i$ exists. If P is $\perp$-complete and (S3) for each family $\{p_i\}_{i\in I}$ of mutually orthogonal elements, $\mu(\sup_{i\in I} p_i) = \sum_{i\in I} \mu(p_i)$, then μ is called completely additive.

Now assume that P is an orthoposet and Δ is a convex set of states on P. Let $V(\Delta)$ be the linear span of Δ taken in the set of all real functions on the orthoposet P and $K(\Delta)$ be the cone generated by Δ. If $\|\cdot\|_\Delta$ is the Minkowski functional corresponding to the absolute convex hull of Δ, the

[1]Sektion Mathematik der Ernst-Moritz-Arndt-Universität, DDR-2200 Greifswald, German Democratic Republic.

0020-7748/89/0100-0027$06.00/0

space $(V(\Delta), K(\Delta), \|\cdot\|_\Delta)$ becomes a base-normed ordered vector space with the base Δ. Let $(V^*(\Delta), \|\cdot\|)$ be the norm dual of $(V(\Delta), \|\cdot\|_\Delta)$. By $p \to p_\Delta$ with $p_\Delta(\varphi) = \varphi(p)$ for all $\varphi \in V(\Delta)$ there is given a natural embedding of P into $V^*(\Delta)$. Its image is denoted by P_Δ. If $V^*(\Delta)^+$ is the dual cone of $K(\Delta)$, then $(V^*(\Delta), V^*(\Delta)^+, \|\cdot\|)$ is an order-unit Banach space with the order unit 1_Δ.

It is easy to see that P_Δ is contained in the order interval $[0, 1_\Delta]$. The present paper deals with the following main question: In which cases does the set $\mathrm{Ext}[0, 1_\Delta]$ of the extreme points of $[0, 1_\Delta]$ and P_Δ coincide? This paper continues Keller (submitted). Also see the very interesting results of Cook (1978, 1988) and Rüttimann (1977, 1985; Cook and Rüttiman, 1985).

2. THE ε-HAHN–JORDAN PROPERTY AND STRONG SETS OF STATES

In this section let P be an orthoposet and Δ a convex set of states.

Definition 1. The set Δ fulfills the ε-Hahn–Jordan property if, for each $\varphi \in V(\Delta)$ and each $\varepsilon > 0$, there exist positive real numbers λ_1, λ_2, elements $\mu_1, \mu_2 \in \Delta$, and a $p \in P$ such that $\varphi = \lambda_1\mu_1 - \lambda_2\mu_2$ and $\lambda_1\mu_2(p^\perp), \lambda_2\mu_2(p) < \varepsilon$.

The ε-Hahn–Jordan property was defined and investigated by Cook (1978). The following lemma is an immediate consequence of his results and the Krein–Milman theorem:

Lemma 1. (i) Δ fulfills the ε-Hahn–Jordan property iff$[0, 1_\Delta]$ is equal to the $\sigma(V^*(\Delta), V(\Delta))$-closure $\mathrm{cl}_{\sigma(V^*(\Delta), V(\Delta))}$ (conv P_Δ) of the convex hull of P_Δ. (ii) If Δ fulfills the ε-Hahn–Jordan property, then $\mathrm{Ext}[0, 1_\Delta] \subseteq \mathrm{cl}_{\sigma(V^*(\Delta), V(\Delta))}(P_\Delta)$.

Recall that Δ is said to be strong if, for all $p, q \in P$, $\{\mu \in \Delta | \mu(p) = 0\} \supseteq \{\mu \in \Delta | \mu(q) = 0\}$ implies $p \leq q$.

Theorem 1. If P is an orthoposet and Δ a strong convex set of states on P such that $P_\Delta = \mathrm{Ext}[0, 1_\Delta]$, then (i) P is a $^\perp$-complete orthomodular poset; (ii) fulfills the ε-Hahn–Jordan property; and (iii) each $\mu \in \Delta$ is a completely additive state.

Proof. The statements (i) and (iii) are verified in Keller (submitted). Statement (ii) follows from Lemma 1(i) and the Krein–Milman theorem. We want to give a sufficient condition that $P_\Delta = \mathrm{Ext}[0, 1_\Delta]$. For this, by $\sigma(P, \Delta)$ let us denote the weakest topology on P which makes each element of Δ continuous. ■

Proposition 1. Let P be an orthoposet and Δ a strong convex set of states with the ε-Hahn–Jordan property such that P is $\sigma(P, \Delta)$-compact.

Then $P_\Delta = \mathrm{Ext}[0, 1_\Delta]$.

Proof. Since P is $\sigma(P, \Delta)$-compact, we obtain $(V^*(\Delta), V(\Delta))$-compactness of P_Δ. Hence, by Lemma 1(i),

$$\mathrm{Ext}[0, 1_\Delta] \subseteq P_\Delta \tag{$*$}$$

For $p \in P$ let

$$F_p = \{f \in [0, 1_\Delta] | (\forall \mu \in \Delta)(\mu(p) = 0 \to f(\mu) = 0)\}$$
$$\cap \{f \in [0, 1_\Delta] | (\forall \mu \in \Delta)(\mu(p) = 1 \to f(\mu) = 1)$$

Since F_p is a compact face of $[0, 1_\Delta]$, by $(*)$ there exists a $q \in P$ with $q_\Delta \in F_p$.

Let us show that $p = q$, then, by the strongness of Δ, there is a $\mu \in \Delta$ with $\mu(p) = 1$ and $\mu(q) < 1$. Otherwise, if $p \nleq q$, we have a $\mu \in \Delta$ with $\mu(p) = 0$ and $\mu(q) > 0$. In both cases $q_\Delta \notin F_p$, which yields a contradiction. Hence $p = q$. ■

Note that the conditions (i) and (iii) in the formulation of Theorem 1 are consequences of $\sigma(P, \Delta)$-compactness of P in general if P is orthomodular. But the difference between the necessary (resp. sufficient) conditions for $\mathrm{Ext}[0, 1_\Delta] = P_\Delta$ given in Theorem 1 (resp. Proposition 1) if Δ is strong is unclear.

In conclusion, using Lemma 1 and methods of Rüttimann (1977) one obtains the following result.

Proposition 2. Let P be a finite orthoposet. Then $P_\Delta = \mathrm{Ext}[0, 1_\Delta]$ if Δ is ultrafull (Rüttimann, 1977) and has the ε-Hahn–Jordan property.

Note that the contrary implication is not valid in general. But, if P is a finite orthoposet and Δ is $\|\cdot\|_\Delta$-closed, then the ε-Hahn–Jordan property implies the Hahn–Jordan property. Under this condition, $P_\Delta = \mathrm{Ext}[0, 1]_\Delta$ iff Δ is ultrafull, which is essentially the main statement of Rüttimann (1977).

3. MODIFICATIONS OF THE STATE SET

Now let us consider our main question from another point of view: What happens if the state set is changed in some sense?

If Δ_1 and Δ_2 are convex sets of states on an orthoposet P with $\Delta_1 \subseteq \Delta_2$, then the restriction of each element of $V^*(\Delta_2)$ to $V(\Delta_1)$ is an element of $V^*(\Delta_1)$. Let us denote the corresponding restriction map by $R(\Delta_1, \Delta_2)$. It holds that $R(\Delta_1, \Delta_2)(P_{\Delta_2}) = P_{\Delta_1}$. Recall that a state set Δ is said to be σ-convex iff, for each countable set $\{\mu_i\}_{i=1}^\infty \subseteq \Delta$ and each set of positive real numbers $\{\lambda_i\}_{i=1}^\infty$ with $\sum_{i=1}^\infty \lambda_i = 1$, we have $\sum_{i=1}^\infty \lambda_i\mu_i \in \Delta$. By $\tilde{\Delta}$ we denote the σ-convex hull of a state set Δ, which is obviously a convex set of states.

Proposition 3. Let Δ be a convex set of states on an orthoposet P. Then (i) $V(\Delta)$ is $\|\cdot\|_\Delta$-complete if Δ is σ-convex; (ii) $R(\Delta, \tilde{\Delta})$ is an isometric order isomorphism from $(V^*(\tilde{\Delta}), V^*(\tilde{\Delta})^+)$ onto $(V^*(\Delta), V^*(\Delta)^+)$; (iii) $(V(\tilde{\Delta}), \|\cdot\|_{\tilde{\Delta}})$ is the $\|\cdot\|_\Delta$-completion of $V(\Delta)$; and (iv) the restriction of $R(\Delta, \tilde{\Delta})$ to $[0, 1_\Delta]$ is an affine $\sigma(V^*(\tilde{\Delta}), V(\hat{\Delta}) - (V^*(\Delta), V(\Delta))$-homeomorphism onto $[0, 1_{\tilde{\Delta}}]$.

Proof. (i) Let Δ be σ-convex and $(\varphi_i)_{i=0}^\infty$ a Cauchy sequence in $V(\Delta)$. One may assume that $\varphi_i - \varphi_{i-1} \in 1/2^i \operatorname{aco} \Delta$, otherwise one chooses a subsequence with this property. Fix $\eta_i^{(1)}, \eta_i^{(2)} \in K(\Delta)$ $(i = 1, 2, \ldots, \infty)$ with $\|\eta_i^{(1)}\|_\Delta, \|\eta_i^{(2)}\|_\Delta \leq 1/2^i$ and $\varphi_i - \varphi_{i-1} = \eta_i^{(2)} - \eta_i^{(2)}$. Since Δ is σ-convex, it is easy to see that $\sum_{i=1}^\infty \eta_i^{(1)}, \sum_{i=1}^\infty \eta_i^{(2)} \in K(\Delta)$ and hence

$$\lim_{i\to\infty} \varphi_i = \varphi_0 + \sum_{i=1}^\infty \eta_i^{(1)} - \sum_{i=1}^\infty \eta_i^{(2)} \in K(\Delta)$$

(ii) The norm dual $(V^*(\Delta))^*$ of $V^*(\Delta)$ is a base-normed ordered Banach space, in which the base can be given by

$$B = \{\varphi \in (V^*(\Delta))^* | \varphi(1_\Delta) = 1, \ \varphi \text{ positive on } V^*(\Delta)^+\}.$$

For each $\varphi \in B$, by $\mu_\varphi(p) = \varphi(p_\Delta)$ there is given a state μ_φ on P. Let $\Delta_B = \{\mu_\varphi | \varphi \in B\}$. It is not difficult to show that $\tilde{\Delta} \subseteq \Delta_B$. This and the fact that each element of $V(\Delta)$ can be interpreted as a bounded linear functional on $(V^*(\Delta))$ gives an extension of each element of $V^*(\Delta)$ to an element of $V^*(\tilde{\Delta})$. Since $K(\Delta)$ is $\|\cdot\|_{\tilde{\Delta}}$-dense in $K(\tilde{\Delta})$, the above extension is uniquely determinated and keeps positivity. Therefore $R(\Delta, \tilde{\Delta})$ becomes an order isomorphism and maps $[-1_{\tilde{\Delta}}, 1_{\tilde{\Delta}}]$ onto $[-1_\Delta, 1_\Delta]$. The last argument completes the proof of (ii).

(iii) Obviously, $V(\Delta)$ is $\|\cdot\|_{\tilde{\Delta}}$-dense in $V(\tilde{\Delta})$. By (i), $V(\tilde{\Delta})$ is complete. It remains to show that the restriction of $\|\cdot\|_{\tilde{\Delta}}$ to $V(\Delta)$ and $\|\cdot\|_\Delta$ coincide. Indeed, for each $\varphi \in V(\Delta)$.

$$\|\varphi\|_{\tilde{\Delta}} = \sup_{f \in [-1_{\tilde{\Delta}}, 1_{\tilde{\Delta}}]} |f(\varphi)| = \sup_{f \in [-1_{\tilde{\Delta}}, 1_{\tilde{\Delta}}]} R(\Delta, \tilde{\Delta}) f(\varphi) = \sup_{g \in [-1_\Delta, 1_\Delta]} g(\varphi) = \|\varphi\|_\Delta$$

(iv) $R(\Delta, \tilde{\Delta})$ is $\sigma(V^*(\tilde{\Delta}), V(\tilde{\Delta})) - (V^*(\Delta), V(\Delta))$-continuous and, by the $\sigma(V^*(\tilde{\Delta}), V(\tilde{\Delta}))$-compactness of $[0, 1_{\tilde{\Delta}}]$, its restriction to $[0, 1_{\tilde{\Delta}}]$ is a closed mapping, which implies (iv). ■

Corollary 1. Let Δ_1 and Δ_2 be convex sets of states on an orthoposet P and $\tilde{\Delta}_1 = \tilde{\Delta}_2$. Then (i) $P_{\Delta_1} = \operatorname{ext}[0, 1_\Delta]$ iff $P_{\Delta_2} = \operatorname{Ext}[0, 1_{\Delta_2}]$; and (ii) Δ_1 fulfills the ε-Hahn–Jordan property iff Δ_2 fulfills the ε-Hahn–Jordan property. ■

The following proposition will be important for our further investigations.

Proposition 4. Let P be an orthoposet and let Δ_1, Δ_2 be convex sets of states with $\Delta_1 \subseteq \Delta_2$. If Δ_2 fulfills the ε-Hahn–Jordan property and $R = R(\Delta_1, \Delta_2)$, then (i) *$R$ is a $\|\cdot\|_{\Delta_2} - \|\cdot\|_{\Delta_1}$-continuous homomorphism onto $V^*(\Delta_1)$;* (ii) *R is $\sigma(V^*(\Delta_2), V(\Delta_2)) - (V^*(\Delta_1), V(\Delta_1))$-continuous; and* (iii) *$R(V^*(\Delta_2)^+) = V^*(\Delta_1)^+$ and $R([0, 1_{\Delta_2}]) = [0, 1_{\Delta_1}]$.*

Proof. It is not difficult to see that R is a $\|\cdot\|_{\Delta_2} - \|\cdot\|_{\Delta_1}$-continuous homomorphism and (ii) is valid. Let us verify (iii) and the surjectivity of R. Let $K = \text{cl}_{\sigma(V^*(\Delta_2), V(\Delta_2))}(\text{conv } P_{\Delta_2})$. Then $K \subseteq [0, 1_{\Delta_2}]$; hence K is compact, and by (ii), $R(K)$ becomes $\sigma(V^*(\Delta_1), V(\Delta_1))$-compact. Since $P_{\Delta_1} \subseteq R(K)$, by Lemma 1(i) one obtains $[0, 1_{\Delta_1}] \subseteq R(K) \subseteq R([0, 1_{\Delta_2})] \subseteq [0, 1_{\Delta_1}]$. This yields (iii) and that R is surjective. ■

Now let Δ be a strong convex set of completely additive states on a complete orthomodular lattice P. I will explain what is meant by "Δ is expectational" under the above restrictions for P and Δ. The concept of an expectational state set can be given, starting from a general orthoposet and a general convex set of states, but if Δ is strong, this generality is not essential. For more details see Keller (submitted) and Rüttimann (1985). A bounded Varadarajan observable on P is a mapping o from the Borel sets $B(\mathbb{R})$ on the real line $\mathbb{R}$ into P satisfying the following axioms:

1. If $\{A_i\}_{i=1}^{\infty}$ is a family of mutually disjoint sets of $B(\mathbb{R})$, then $o(A_i) \perp o(A_j)$ for $i \neq j$ and $\sup_{i=1}^{\infty} o(A_i) = 0(\bigcup_{i=1}^{\infty} A_i)$.
2. There exists a bounded $A \in B(\mathbb{R})$ with $o(A) = 1$.

By $\mu \to \int id\, d\mu o$ for $\mu \in \Delta$ there is given a bounded affine function on Δ which has a uniquely defined extension to an element $E(o)$ of $V^*(\Delta)$.

Δ is said to be expectational if $V^*(\Delta) = \{E(o) | o$ is a bounded Varadarajan observable$\}$.

Theorem 2. Let P be a complete orthomodular lattice, Δ_2 an expectational, strong convex state set, and Δ_1 a unital (Cook, 1978) convex set of states fulfilling the ε-Hahn–Jordan property. If $\Delta_1 \subseteq \Delta_2$, then $R(\Delta_2, \Delta_2)$ is an isometric order isomorphism and $(V(\tilde{\Delta}_1), \|\cdot\|_{\tilde{\Delta}_1}) = (V(\tilde{\Delta}_2), \|\cdot\|_{\tilde{\Delta}_2})$

Proof. By Proposition 4(i), $R = R(\Delta_1, \Delta_2)$ is a homomorphism from $V^*(\Delta_2)$ onto $V^*(\Delta_1)$. If o is a Varadarajon observable with $E(o) \neq 0$, then $o(]0, \infty[) \neq 0$ or $o(]-\infty, 0[) \neq 0$. One may assume the first statement. Since Δ_1 is unital, there exists a $\mu \in \Delta_1$ with $\mu o(]0, \infty[) = 1$. Using this fact, one easily shows $E(o)(\mu) \neq 0$, hence $R(E(o))(\mu) \neq 0$. This and Proposition 4(iii) imply that R is an isometric order isomorphism and, by Proposition 3(ii), that $R(\tilde{\Delta}_1, \tilde{\Delta}_2)$ is an isometric order isomorphism, too. Therefore, by Proposition 3(i), $V(\tilde{\Delta}_1)$ is a closed subspace of the Banach space $(V(\tilde{\Delta}_2), \|\cdot\|_{\tilde{\Delta}})$. Since each element of $V^*(\tilde{\Delta}_1)$ has only one extension to an element of $V^*(\tilde{\Delta}_2), (V(\tilde{\Delta}_1), \|\cdot\|_{\tilde{\Delta}_1}) = (V(\tilde{\Delta}_2), \|\cdot\|_{\tilde{\Delta}_2})$. ■

For a convex set Δ of states on an orthoposet P let $\tilde{\Delta} = \{\varphi \in V(\Delta) | \varphi$ is a state$\}$.

Lemma 2. If Δ fulfills the ε-Hahn–Jordan property, then (i) $\mathrm{cl}_{\|\cdot\|_\Delta}(\Delta) = \bar{\Delta}$ and (ii) $\|\cdot\|_\Delta = \|\cdot\|_{\bar{\Delta}}$ and $V^*(\Delta)^+ = V^*(\bar{\Delta})^+$.

Proof. (i) Let $(V^*(\Delta))^*$ be the norm dual of $V^*(\Delta)$ and $\widetilde{K(\Delta)}$ be the set of all positive elements in $(V^*(\Delta))^*$. One may assume $V(\Delta)$ to be a subspace of $(V^*(\Delta))^*$ in the usual fashion. It is easy to see that $\widetilde{K(\Delta)} \cap V(\Delta)$ is $\|\cdot\|_\Delta$-closed and contains $K(\Delta)$; moreover,

$$\widetilde{K(\Delta)} \cap (\Delta) = \mathrm{cl}_{\|\cdot\|_\Delta}(K(\Delta)) \qquad (*)$$

Obviously, each $\varphi \in \mathrm{cl}_{\|\cdot\|_\Delta}(\Delta)$ is a state on P, hence it is contained in $\bar{\Delta}$.

In the other direction, if $\varphi \in \bar{\Delta}$, then by Lemma 1(i), φ can be considered as an element of $\overline{K(\Delta)}$; hence, by $(*)$, $\varphi \in \mathrm{cl}_{\|\cdot\|_\Delta}(K(\Delta))$. Let $(\varphi_i)_{i=1}^\infty$ be a sequence in $\widetilde{K(\Delta)} \backslash \{0\}$ with $\lim_{i\to\infty} \varphi_i = \varphi$. Then $\lim_{i\to\infty} \varphi_i / \|\varphi_i\|_\Delta = \varphi / \|\varphi\|_\Delta$, which shows $\varphi \backslash \|\varphi\|_\Delta \in \mathrm{cl}_{\|\cdot\|_\Delta}(\Delta)$; hence $\varphi(1)/\|\varphi\|_\Delta = 1$. Since φ is a state, $\varphi(1) = 1$. Therefore, $\varphi = \varphi/\|\varphi\|_\Delta \in \mathrm{cl}_{\|\cdot\|_\Delta}(\Delta)$.

(ii) Now $\bar{\Delta} \subseteq B_\Delta$, where B_Δ is the closed unit ball in $(V(\Delta), \|\cdot\|_\Delta)$. This implies aco $\Delta \subseteq$ aco $\bar{\Delta} \subseteq \mathrm{B}_\Delta$, and hence $\|\cdot\|_\Delta = \|\cdot\|_{\bar{\Delta}}$. Then $V^*(\Delta)^+ = V^*(\bar{\Delta})^+$ is obvious. ■

Corollary 2. Let P be the projection lattice of a W^*-algebra without any direct summand of type I_2. Then the set of all completely additive states is the only strong, $\|\cdot\|_\Delta$-closed, and σ-convex state set Δ on P with $P_\Delta = \mathrm{Ext}[0, 1_\Delta]$.

Proof. Let Δ_2 be the set of all completely additive states on P and Δ_1 a strong, $\|\cdot\|_{\Delta_2}$-closed, and σ-convex state set with $P_{\Delta_1} = [0, 1_{\Delta_1}]$. Then, by Theorem 1, $\Delta_1 \subseteq \Delta_2$ and Δ_1 has the ε-Hahn–Jordan property. Δ_2 is $\|\cdot\|_{\Delta_2}$-closed, σ-convex, and expectational by the Gleason–Christensen–Yeadon theorem (Yeadon, 1984). Using Theorem 2, one obtains $(V(\Delta_1), \|\cdot\|_{\Delta_1}) = (V(\Delta_2), \|\cdot\|_{\Delta_2})$. By Lemma 2 it follows that $\Delta_1 = \Delta_2$. ■

Finally, we want to characterize Boolean algebras under the view of representability by extreme point posets in the above manner. Recall that Boolean algebras can be considered as distributive (orthomodular) ortholattices. Dixmier has introduced the concept of a hyperstonean compact topological T_2 space. A Boolean algebra is said to be hyperstonean if its Stonean representation space is a hyperstonean compact topological T_2 space. With regard to a paper of Flachsmeyer (1979), let us give a characterization of hyperstonean Boolean algebras:

A complete Boolean algebra is hyperstonean iff it admits a unital (convex) set of completely additive states.

Definition 2. Let H be a complex Hilbert space and $(\mathrm{Hilb}(H), \vee, \wedge, {}^\perp, \{0\}, H)$ the complete orthomodular lattice of the closed linear subspaces of H, where $^\perp$ is the usual orthocomplementation. Let us call a $^\perp$-closed regular sublattice of $\mathrm{Hilb}(H)$ an H-lattice. Here a sublattice L' of a given complete lattice L is called regular if L' contains the supremum of each subset of L'.

It is clear that each H-lattice is a complete orthomodular lattice itself.

Proposition 5. A Boolean algebra is hyperstonean iff there exists a complex Hilbert space H such that P is isomorphic to a Boolean H-lattice.

Proof. Obviously, since $\mathrm{Hilb}(H)$ admits a strong convex set of completely additive states, each Boolean H-lattice becomes a hyperstonean Boolean algebra.

If P is a hyperstonean Boolean algebra, then it can be considered as the projection lattice of a commutative W^*-algebra A (Takesaki, 1979). Let (π, H) be a normal representation of A and $B(H)$ the algebra of bounded linear operators on H. Then $\pi(A)$ is a von Neumann algebra. We identify $\mathrm{Hilb}(H)$ with the orthomodular lattice $P(B(H))$ of the orthogonal projections in $B(H)$. Since π_{1_P} is a $^\perp$-order isomorphism from P into $P(B(H))$, it remains to show that $\pi(P)$ is a regular sublattice of $P(B(H))$.

If $\{p_i\}_{i\in I}$ is a family of mutually orthogonal elements of P, then by Theorem 2.8.4 of Pedersen (1979)

$$\bigvee_{i\in I} \pi(p_i) \in \pi(P) \qquad (*)$$

By transfinite induction and a lemma of Iwamura (1944) on directed nets, one obtains $\bigvee_{i\in I} \pi(p_i) \in \pi(P)$ for each upward-directed net $(p_i)_{i\in I}$ in P. Hence we only have to verify that $\pi(p) \wedge \pi(q) \in \pi(P)$ for all $p, q \in P$. Indeed, for $p, q \in P$,

$$p \vee q = (p \wedge q^\perp) \vee (p \wedge q) \vee (p^\perp \wedge q) \qquad (**)$$

and $p \wedge q^\perp$, $p \wedge q$, and $p^\perp \wedge q$ are mutually orthogonal. Hence, by $(*)$ there exists an $r \in P$ with $\pi(r) = \pi(p \wedge q^\perp) \vee \pi(p \wedge q) \vee \pi(p^\perp \wedge q)$. Assume that $\pi(r) < \pi(p \vee q)$. Then $r < p \vee q$ and there exists an $s \in P\backslash\{0\}$ with $s \le p \vee q$ and $s \perp r$, i.e., $r \le s^\perp$. This implies $p \wedge q^\perp$, $p \wedge q$, $p^\perp \wedge q \le s^\perp$, hence $p \wedge q^\perp$, $p \wedge q$, $p^\perp \wedge q \perp s$, which contradicts $(**)$.

Therefore

$$\pi(r) = \pi(p \vee q) \ge \pi(p) \vee \pi(q) \ge \pi(p \wedge q^\perp), \pi(p \wedge q), \pi(p^\perp \wedge q),$$

and we obtain $\pi(p) \vee \pi(q) = \pi(r) \in \pi(P)$. ■

Now we give a characterization of Boolean algebras with regard to our main question.

Proposition 6. Let P be an orthoposet and Δ a strong convex set of states on P with $P_\Delta = \mathrm{Ext}[0, 1_\Delta]$.

Then P is a Boolean algebra iff $(V^*(\Delta), V^*(\Delta)^+)$ is a lattice.

Proof. If P is a Boolean algebra, then, by Theorem 1, it becomes hyperstonean. Hence P can be considered to be the projection lattice of a commutative W^*-algebra A. By Corollary 1, $P_{\tilde{\Delta}} = \mathrm{Ext}[0, 1_{\tilde{\Delta}}]$ and then, by Lemma 2,

$$P_{\mathrm{cl}_{\tilde{\Delta}}(\tilde{\Delta})} = \mathrm{Ext}[0, 1_{\mathrm{cl}_{\tilde{\Delta}}(\tilde{\Delta})}]$$

Furthermore, by Lemma 2, $\Delta^* = \mathrm{cl}_{\tilde{\Delta}}(\tilde{\Delta})$ is a $\|\cdot\|_{\Delta^*}$-closed, strong, and σ-convex set of states on P. This and Corollary 2 imply that Δ^* is the set of all completely additive states—say, normal measures—on P. Applying well-known facts of functional analysis, we obtain $(V^*(\Delta^*), V^*(\Delta^*)^+)$ is a lattice, and by Proposition 3(ii) and Lemma 2(ii), $(V^*(\Delta), V^*(\Delta)^+)$, too.

Let us verify the other implication. $(\mathrm{Ext}[0, 1_\Delta], \leq, {}^\perp, 0, 1_\Delta)$ with $x^\perp = 1_\Delta - x$ $(x \in \mathrm{Ext}[0, 1_\Delta])$ is an orthoposet, which becomes a Boolean algebra if $V^*(\Delta)$ is a lattice (Keller, to appear). It is easy to show that $p \to p_\Delta$ is an ${}^\perp$-order isomorphism from P onto $\mathrm{Ext}[0, 1_\Delta]$, which completes the proof.

In conclusion, see Cook (1988) on the natural embedding $p \to p_\Delta$ if $V(\Delta)$ is a lattice. ■

REFERENCES

Cook, T. (1978). The geometry of generalized quantum logics, *International Journal of Theoretical Physics*, **17**, 941-955.

Cook, T. (1988). Riesz spaces and quantum logics, preprint.

Cook, T., and Rüttimann, G. T. (1985). Symmetries on quantum logics, *Reports on Mathematical Physics*, **21**, 121-126.

Flachsmeyer, J. (1979). Underlying Boolean algebras, *Mathematical Centre Tracts*, **115**, 91-103.

Iwamura, T. (1944). A lemma on directed sets, *Zenkoku Shijo Sugaku Danwakai*, **262**, 107-111.

Keller, K. (to appear). Orthoposets of extreme points of order intervals, *Mathematische Nachrichten*.

Keller, K. (submitted). Orthoposets of extreme points and quantum logics, *Reports on Mathematical Physics*.

Narici, L., and Beckenstein, E. (1985). *Topological Vector Spaces*, Marcel Dekker, New York.

Ng, K.-F. and Wong, Y.-C. (1973). *Partially Ordered Topological Vector Spaces*, Clarendon Press, Oxford.

Pedersen, G. K. (1979). *C*-Algebras and Their Automorphism Groups*, Academic Press, London.

Rüttimann, G. T. (1977). Jordan-Hahn decomposition of signed weights of finite orthogonality spaces, *Commentarii Mathematici Helvetici*, **52**, 129-144.

Rüttimann, G. T. (1985). Expectation functionals of observables and counters, *Reports on Mathematical Physics*, **21**, 213-222.

Takesaki, M. (1979). *Theory of Operator Algebras* I, Springer-Verlag, New York.

Yeadon, F. J. (1984). Finitely additive measures on projections in finite W^*-algebras, *Bulletin of the London Mathematical Society*, **16**, 145-150.